Random Walk in Random Environment Chapter 4

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1 $G(x) = H(x)$

We give the proof about $G(x) = H(x)$.

$$\int_{-x}^{x} \sum_{k=-\infty}^{+\infty} (-1)^k \exp\left(-\frac{1}{2}(u-2kx)^2\right) du = \sum_{k=-\infty}^{+\infty} (-1)^k \int_{(2k-1)x}^{(2k+1)x} e^{-\frac{1}{2}u^2}\, du$$
$$= \int_{-\infty}^{+\infty} \sum_{k=-\infty}^{+\infty} (-1)^k \mathbf{1}_{[(2k-1)x,(2k+1)x]}(u) e^{-\frac{1}{2}u^2}\, du.$$

It is obvious that $\sum_{k=-\infty}^{+\infty} (-1)^k \mathbf{1}_{((2k-1)x,(2k+1)x)}(u)$ is a $4x$-periodic function and is even(consider function graph).

$$\sum_{k=-\infty}^{+\infty} (-1)^k \mathbf{1}_{((2k-1)x,(2k+1)x)}(u) = a_0 + \sum_{n=1}^{+\infty} a_n \cos\left(\frac{2n\pi}{4x}u\right),$$

where

$$a_0 = \frac{1}{4x}\int_{-2x}^{2x} \left(-\mathbf{1}_{(-2x,-x)}(u) + \mathbf{1}_{(-x,x)}(u) - \mathbf{1}_{(x,2x)}(u)\right) du = 0$$

and

$$a_n = \frac{2}{4x}\int_{-2x}^{2x} \left(-\mathbf{1}_{(-2x,-x)}(u) + \mathbf{1}_{(-x,x)}(u) - \mathbf{1}_{(x,2x)}(u)\right) \cdot \cos\left(\frac{2n\pi}{4x}u\right) du$$
$$= \frac{4}{n\pi}\sin\left(\frac{1}{2}n\pi\right).$$

Therefore,

$$\sum_{k=-\infty}^{+\infty} (-1)^k \mathbf{1}_{((2k-1)x,(2k+1)x)}(u) = \sum_{n=1}^{+\infty} \frac{4}{n\pi}\sin\left(\frac{1}{2}n\pi\right)\cos\left(\frac{2n\pi}{4x}u\right)$$
$$= \frac{4}{\pi}\sum_{k=0}^{+\infty} \frac{1}{n}\sin\left(\frac{1}{2}(2k+1)\pi\right)\cos\left(\frac{(2k+1)\pi}{2x}u\right)$$
$$= \frac{4}{\pi}\sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1}\cos\left(\frac{(2k+1)\pi}{2x}u\right).$$

Now, we have

$$\begin{aligned}
&\frac{1}{\sqrt{2\pi}}\int_{-x}^{x}\sum_{k=-\infty}^{+\infty}(-1)^k\exp\left(-\frac{1}{2}(u-2kx)^2\right)du\\
=&\int_{-\infty}^{+\infty}\frac{4}{\pi}\sum_{k=0}^{\infty}\frac{(-1)^k}{2k+1}\cos\left(\frac{(2k+1)\pi}{2x}u\right)\cdot\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}u^2}\,du\\
=&\mathbb{E}\left[\frac{4}{\pi}\sum_{k=0}^{\infty}\frac{(-1)^k}{2k+1}\cos\left(\frac{(2k+1)\pi}{2x}\mathbf{X}\right)\right]\\
=&\frac{4}{\pi}\sum_{k=0}^{\infty}\frac{(-1)^k}{2k+1}\mathbb{E}\left[\cos\left(\frac{(2k+1)\pi}{2x}\mathbf{X}\right)\right]
\end{aligned}$$

where $\mathbf{X}\sim N(0,1)$. Note that

$$\begin{aligned}
&\mathbb{E}\left[\cos\left(\frac{(2k+1)\pi}{2x}\mathbf{X}\right)+i\sin\left(\frac{(2k+1)\pi}{2x}\mathbf{X}\right)\right]\\
&=\mathbb{E}\left[\exp\left(i\frac{(2k+1)\pi}{2x}\mathbf{X}\right)\right]=\exp\left(-\frac{(2k+1).\pi^2}{8x^2}\right)
\end{aligned}$$

We proved that $G(x)=H(x)$.

2 Recurrence of random walk on $\mathbb{Z}$ by optional stopping

Theorem 2.1 (Optional stopping theorem). *Let $\{M_n\}_{n\in\mathbb{N}}$ be a martingale and T is a stopping time respect to filtration $\{\mathscr{F}_n\}_{n\in\mathbb{N}}$. We have that $\mathbb{E}[|M_T|]<\infty$ and $\mathbb{E}M_T=\mathbb{E}[M_0]$ if one of the following holds*

(i) The stopping time T is a.s. bounded; that is, there exists $C\geq 0$ such that $T\leq C$ a.s..

(ii) $T<\infty$ a.s. and $\{M_n\}_{n\in\mathbb{N}}$ is uniformly integrable and $\mathbb{E}[|M_T|]<\infty$.

Remark 2.2. *Note that the condition $\mathbb{E}[|M_T|]<\infty$ is redundant. Indeed, by the martingale convergence theorem we know that if $\{M_n\}_{n\in\mathbb{N}}$ is a martingale with $\sup_n\mathbb{E}[|M_n|]<\infty$, then there exists a random variable M_∞ such that $M_n\to M_\infty$ a.s. and $\mathbb{E}[|M_\infty|]<\infty$. Let $\mathbf{M}_n=M_{T\wedge n}$, then we have*

$$\sup_n\mathbb{E}[|\mathbf{M}_n|]\leq\sup_n\mathbb{E}[|M_{T\wedge n}|]\leq\sup_n\mathbb{E}[|M_n|]<\infty.$$

In fact, $\mathbb{E}[M_{T\wedge n}]\leq\mathbb{E}[M_n]$ for $T<\infty$ a.s.. It is obvious that $\{|M_n|\}_{n\in\mathbb{N}}$ is a sub-martingale, define $U_n=|M_n|-|M_{T\wedge n}|$, we can obtain that

$$U_{n+1}-U_n=(|M_{n+1}|-|M_n|)\mathbf{1}_{\{T\leq n\}}$$

due to $\left|M_{T\wedge(n+1)}\right|-|M_{T\wedge n}|=(M_{n+1}-M_n)\mathbf{1}_{\{T>n\}}$. Therefore,

$$\mathbb{E}[U_{n+1}-U_n]=\mathbf{1}_{\{T\leq n\}}\cdot\mathbb{E}[|M_{n+1}|-|M_n|\,|\mathscr{F}_n]\geq 0.$$

It is obvious that $\{\mathbf{M}_n\}_n=\{M_{T\wedge n}\}_n$ is martingale, therefore, we have

$$\mathbf{M}_n\to\mathbf{M}_\infty \text{ a.s..}$$

and $\mathbb{E}[|\mathbf{M}_\infty|]<\infty$(i.e. $\mathbb{E}[M_T]<\infty$ since $T<\infty$ a.s.).

Remark 2.3. *The Optional stopping theorem is important, Let $T=\inf\{n:S_n=1\}$, we can prove that $T<\infty$ a.s.(recurrent), but $S_T=1$ a.s., $S_0=0$, we observe that $\mathbb{E}[S_T]\neq\mathbb{E}[S_0]$.*

It is obviously that $\{S_n\}_{n\in\mathbb{N}_+}$ is a martingale. Let $T_z := \inf\{n : S_n = z\}$, then T_z is a stopping time. For $a < 0 < b$, define stop time $T_{a,b} := T_a \wedge T_b$, which is the first exit time of (a, b). Since $M_n := S_{T_{a,b}\wedge n} \leq |a| \vee |b|$ $a.s.$, and $T_{a,b} < \infty$ $\mathbb{P}_0$-$a.s.$, we have

$$0 = \mathbb{E}[M_{T_{a,b}}] = \mathbb{P}(T_a < T_b) \cdot a + \mathbb{P}(T_b < T_a) \cdot b = \mathbb{P}(T_a < T_b)(a - b) + b.$$

(where $T_{a,b} < \infty$ $a.s.$ is due to that Wald's identities and Dominated convergence theorem) can obtain $\mathbb{E}[T_{a,b}] < \infty$. Therefore,

$$\mathbb{P}_0(T_a < T_b) = \frac{b}{b - a}.$$

Note that $i + S_n$ have the distribution of a random walk started at i, Thus, for all $0 \leq i \leq k$,

$$\mathbb{P}_i(T_0 < T_k) = \mathbb{P}_0(T_{-i} < T_{k-i}) = \frac{k - i}{k}.$$

Note that

$$\mathbb{P}_i(T_0 = \infty) = \lim_{n\to\infty} \mathbb{P}_i(T_n < T_0) = 0,$$

Remark 2.4. *We also can prove that $T_{a,b} < \infty$ a.s. by estimating $\mathbb{P}(T_{a,b} > nI)$ where $I := b-a$. In fact, for any $a < x < b$,*

$$\mathbb{P}(S_{n+I} \notin (a, b)|S_n = x, T_{a,b} > n) \geq \mathbb{P}(\forall\ 0 \leq j < I, X_{n+j+1} = 1|S_n = x, T_{a,b} > n) = 2^{-I}.$$

Since $T_{a,b} > n + I$ implies that $T_{a,b} > n$ and $S_{n+I} \in (a, b)$, therefore, we have

$$\begin{aligned}
&\mathbb{P}(T_{a,b} > n + I) \\
=&\sum_{x=a+1}^{b-1} \mathbb{P}(T_{a,b} > n + I|S_n = x, T_{a,b} > t) \cdot \mathbb{P}(S_n = x, T_{a,b} > n) \\
\leq&(1 - 2^{-I}) \cdot \mathbb{P}(T_{a,b} > n)
\end{aligned}$$

Inductively, we have

$$\mathbb{P}(T_{a,b} > nI) \leq (1 - 2^{-I})^n.$$

3 Borel-Cantelli Lemma and almost sure convenience

The proofs of almost all strong theorem are based on different forms of the Borel-Cantelli Lemma and those of the Markov inequality. *The main idea of Borel-Cantelli Lemma is to construct a series to control the probability of evens.*

Lemma 3.1. *If $\sum_{n=1}^{\infty} \mathbb{P}(A_n) < \infty$, then $\mathbb{P}(\limsup_{n\to\infty} A_n) := \mathbb{P}(A_n\ i.o.) = 0$.*

Proof. Define general r.v. $\xi := \sum_{n=1}^{\infty} \mathbf{1}_{A_n}$, it is obvious ξ is not negative. By $\mathbb{E}[\xi] = \sum_{n=1}^{\infty} \mathbb{P}(A_n) < \infty$, we have $\xi < \infty$ a.s., which is due to

$$\mathbb{P}(\xi = \infty) \leq \mathbb{P}(\xi \geq N) \leq \frac{1}{N}\mathbb{E}[\xi].$$

By $\xi < \infty$ a.s., we have $\mathbb{P}(A_n\ i.o.) = 0$. ☐

Proof.

$$\mathbb{P}(\limsup_{n\to\infty} A_n) = \lim_{n\to\infty} \mathbb{P}(\cup_{k\geq n} A_k) \leq \lim_{n\to\infty} \sum_{k=n}^{\infty} \mathbb{P}(A_k) = 0.$$

☐

Corollary 3.2. *If*

(i) $\sum_{n=1}^{\infty} \mathbb{P}(A_n|B_n) < \infty$,

(ii) B_n *occurs a.s. if* n *is large enough,*

then A_n *occurs a.s. only finitely many times.*

Proof. By Lemma 3.1, we have

$$\sum_{n=1}^{\infty} \mathbb{P}(A_n \cap B_n) \leq \sum_{n=1}^{\infty} \mathbb{P}(A_n|B_n)\mathbb{P}(B_n) \leq \sum_{n=1}^{\infty} \mathbb{P}(A_n|B_n).$$

Therefore, A_nB_n occurs a.s. finitely many times. By (ii), we complete the proof since $B_n = \Omega$ if n is large enough. □

The converse of the Borel-Cantelli lemma is trivially false.

Example 3.3. Let $\Omega = (0,1)$, $\mathcal{F} = \mathscr{B}((0,1))$ and $\mathbb{P} = \lambda$. If $A_n = (0, a_n)$ where $a_n \to 0$ as $n \to \infty$, then $\limsup A_n = \emptyset$, but if $a_n = \frac{1}{n}$, we have $\sum a_n = \infty$.

Lemma 3.4. *Let* $S_n := \sum_{k=1}^{n} X_k$*, where* $X_k \geq 0$*. If* $\mathbb{E}[S_n] \to \infty$*,* $\sup_{n\geq 1} \mathbb{E}[X_n] < \infty$ *and we can find* $C, \delta > 0$ *such that for any* $n \in \mathbb{N}_+$*,*

$$\mathbf{Var}(S_n) \leq C \cdot (\mathbb{E}[S_n])^{2-\delta} \tag{1}$$

then

$$\lim_{n\to\infty} \frac{S_n}{\mathbb{E}[S_n]} = 1 \text{ a.s.}.$$

Proof. We can assume $0 < M := \sup_{n\geq 1} \mathbb{E}[X_n] \leq 1$. Note that $0 \leq \mathbb{E}[X_n] \leq 1$ and $\mathbb{E}[S_n] \to \infty$, it is easy to see the integer part of $\{E(n) := \mathbb{E}[S_n]\}_{n\geq 1}$ can take all natural numbers. Therefore, we can find a subsequence $\{n_k\}_{k\geq 1}$, such that

$$k^{\frac{2}{\delta}} \leq E(n_k) \leq k^{\frac{2}{\delta}} + 1, \quad \forall\, k \geq 1.$$

By Markov's inequality, and (1), we have

$$\mathbb{P}\left(\left|\frac{S_{n_k}}{E(n_k)} - 1\right| \geq \varepsilon\right) \leq \frac{\mathbf{Var}(S_{n_k})}{\varepsilon^2 \cdot E(n_k)^2} \leq \frac{C}{\varepsilon^2 \cdot k^2}, \quad \forall k \geq 1, \varepsilon > 0.$$

By Borel-Cantelli's lemma, we have

$$\lim_{k\to\infty} \frac{S_{n_k}}{E(n_k)} = 1 \text{ a.s.}.$$

For n large enough, there exista k large enough such that $n \in [n_k, n_{k+1})$. In this time, utilize the monotonicity of S_n and $E(n)$, we have

$$\frac{E(n_k)}{E(n_{k+1})} \cdot \frac{S_{n_k}}{E(n_k)} \leq \frac{S_n}{E(n)} \leq \frac{E(n_{k+1})}{E(n_k)} \cdot \frac{S_{n_{k+1}}}{E(n_{k+1})}.$$

Since $\frac{E(n_{k+1})}{E(n_k)} \to 1$ when $k \to \infty$, we complete the proof. □

Lemma 3.5. *If $\{A_n\}_{n\geq 1}$ are independent evens, then*

$$\sum_{n=1}^{\infty}\mathbb{P}(A_n)=\infty\Rightarrow\mathbb{P}(A_n\ i.o.)=1.$$

Proof. Since $\mathbb{P}(\liminf_{n\to\infty}A_n)=\lim_{n\to}\mathbb{P}(\cup_{k\geq n}A_k^c)$, by the independence of $\{A_n\}_{n\geq 1}$, we have

$$\mathbb{P}(\cap_{k=n}^m A_k^c)=\Pi_{k=n}^m\mathbb{P}(A_k^c)=(1-\mathbb{P}(A_k))\leq\Pi_{k=n}^m\exp(-\mathbb{P}(A_k))=\exp\left(-\sum_{k=n}^m\mathbb{P}(A_k)\right)\to 0(m\to\infty).$$

Therefore,

$$\mathbb{P}(\liminf_n A_n^c)=\lim_{n\to\infty}\mathbb{P}(\cap_{k=n}^{\infty}A_k^c)=\lim_{n\to\infty}\lim_{m\to\infty}\mathbb{P}(\cap_{k=n}^m)=0.$$

□

In the following, we use Corollary 3.4 to prove

Lemma 3.6. *If $\{A_n\}_{n\geq 1}$ are pairwise independent evens, then*

$$\sum_{n=1}^{\infty}\mathbb{P}(A_n)=\infty\Rightarrow\mathbb{P}(A_n\ i.o.)=1.$$

Proof. Let $S_n:=\sum_{k=1}^n\mathbf{1}_{A_k}$, we compute the variation of S_n, for any $n\in\mathbb{N}_+$,

$$\begin{aligned}\mathbf{Var}(S_n)&=\sum_{k=1}^n\mathbf{Var}(\mathbf{1}_{A_k})+2\sum_{1\leq i<j\leq n}\mathbf{Cov}(\mathbf{1}_{A_i},\mathbf{1}_{A_j})\\&=\sum_{k=1}^n\mathbb{P}(A_k)-\sum_{k=1}^n\mathbb{P}^2(A_k)\leq\mathbb{E}[S_n].\end{aligned}$$

Due to $\mathbb{E}[S_n]\to\infty$, by Corollary 3.4, we have $S_n\to\infty$ a.s.. □

Proof. We only need to prove $\mathbb{P}(S_\infty\leq a)=0\ \forall\, a>0$. For any $a>0$, take $N\geq 1$ large enough, such that $\mathbb{E}[S_N]\geq a$. Then for any $n\geq N$, we have

$$\begin{aligned}\mathbb{P}(S_\infty\leq a)&\leq\mathbb{P}(S_n\leq a)\\&\leq\mathbb{P}(-(S_n-\mathbb{E}[S_n])\geq\mathbb{E}[S_n]-a)\\&\leq\frac{\mathbb{E}[|S_n-\mathbb{E}[S_n]|^2]}{|\mathbb{E}[S_n]-a|^2}=\frac{\mathbf{Var}(S_n)}{|\mathbb{E}[S_n]-a|^2}\\&\leq\frac{\mathbb{E}[S_n]}{|\mathbb{E}[S_n]-a|^2}\to 0,\end{aligned}$$

when $n\to\infty$, since $\mathbb{E}[S_n]\to\infty$. □

Lemma 3.7. *Let $A_1, A_2, cdots$ be a sequence of evens for which*

$$\sum_{n=1}^{\infty}\mathbb{P}(A_n)=\infty,$$

and

$$\liminf_{n\to\infty}\frac{\sum_{k=1}^n\sum_{i=1}^n\mathbb{P}(A_kA_i)}{(\sum_{k=1}^n\mathbb{P}(A_k))^2}\leq C\quad(C\geq)$$

then

$$\mathbb{P}(\limsup_{n\to A_n})\geq C^{-1}.$$

The ideas to prove a.s. convergence by Borel-Cantelli lemma:

Lemma 3.8. *Let $\{T_n\}_{n\geq 1}$ be r.v. such that*

$$\sum_{n=1}^{\infty} \mathbb{P}(|T_n| > \varepsilon) < \infty$$

for each $\varepsilon > 0$. Then $T_n \to 0$ a.s..

Proof. For each $k \geq 1$,

$$\sum_{n=1}^{\infty} \mathbb{P}(|T_n| > 2^{-k}) < \infty.$$

Hence, by the Borel-Cantelli lemma(use $[\limsup_n A_n]^c$), for each $k \geq 1$, $|T_n| \leq 2^{-k}$ for all n sufficiently large, except on a null event N_k. It follows that

$$T_n(\omega) \to 0 \quad \text{for all } \omega \notin \cup_{k=1}^{\infty} N_k.$$

Since $\cup_{k=1}^{\infty} N_k$ is a null event, $T_n \to 0$ a.s. follows. □

Lemma 3.9. *Let $\{T_n\}_{n\geq 1}$ be r.v. such that*

$$\sum_{n=1}^{\infty} \mathbb{P}(|T_n| > \varepsilon_n) < \infty$$

for positive constant $\varepsilon_n \to 0$. Then $T_n \to 0$ a.s..

Proof. Applying Borel-Cantelli lemma to events $\{|T_n| > \varepsilon_n\}, n \geq 1$. □

We need to estimate $\mathbb{P}(|T_n| > \varepsilon)$, it just the Chebyshev inequality

$$\mathbb{P}(|X| > \varepsilon) \leq \mathbb{E}[|X|]/\varepsilon$$

$$\mathbb{P}(|X - \mathbb{E}[X]| > \varepsilon) \leq \mathbf{Var}(X)/\varepsilon^2$$

and

$$\mathbb{P}(X > \varepsilon) \leq \exp(-t\varepsilon)\mathbb{E}[\exp(tX)]$$

for each $\varepsilon > 0$ and real t.

The ideas to prove a.s. convergence. The moment estimate also is useful.

Lemma 3.10. *Suppose that*

$$\sum_{n=1}^{\infty} \mathbb{E}[|T_n|^p] < \infty,$$

for some $p > 0$, then $T_n \to 0$ a.s..

Proof. By $\sum_{n=1}^{\infty} \mathbb{E}[|T_n|^p] < \infty$, we have $\mathbb{E}\left[\sum_{n=1}^{\infty} |T_n|^p\right] < \infty$. Moreover, $\sum_{n=1}^{\infty} |T_n|^p < \infty$ a.s. and hence that $T_n \to 0$ a.s..

□

The ideas to prove a.s. convergence by extracting subsequence.

Lemma 3.11. *Let $\{X_n\}_{n\geq 1}$ is r.v. sequence, if there exists a subsequence $\{n_k\}_{k\geq 1}$ such that*

$$X_{n_k} \to X \ a.s..$$

and

$$\max_{n_{k-1}<n\leq n_k} \left|X_n - X_{n_{k-1}}\right| \to 0 \ a.s..$$

Then,

$$X_n \to X \ a.s..$$

Proof. For n large enough, there exist a unique k such $n_{k-1} < n \leq n_k$, then

$$|X_n - X| \leq \left|X_{n_{k-1}} - X\right| + \max_{n_{k-1}<m\leq n_k} \left|X_m - X_{n_{k-1}}\right| \to 0 \ a.s..$$

□

Theorem 3.12.

$$\lim_{n\to\infty} \frac{S_n}{n} = 0 \quad a.s.. \tag{2}$$

Proof. Since $\mathbb{E}[\frac{S_n}{n}] = 0$ and $\mathbb{E}[\frac{S_n^2}{n^2}] = \frac{1}{n}$, by Chebyshev inequality, for any $\varepsilon > 0$, we have

$$\mathbb{P}\left(\left|\frac{S_n}{n}\right| \geq \varepsilon\right) \leq \frac{1}{n\varepsilon^2},$$

Therefore, we have $\frac{S_{n^2}}{n^2} \to 0$ a.s. when $n \to \infty$. Now we have to estimate the value of S_k for the k lying in the gap. If $n^2 \leq k < (n+1)^2$, then

$$\begin{aligned}\left|\frac{S_k}{k}\right| &= \left|\frac{S_{n^2}n^2}{kn^2} + \frac{S_k - S_{n^2}}{k}\right| \\ &\leq \left|\frac{S_{n^2}}{n^2}\right| + \left|\frac{k-n^2}{k}\right| \leq \left|\frac{S_{n^2}}{n^2}\right| + \left|\frac{(n+1)^2-n^2}{k}\right| \to 0 \quad a.s..\end{aligned}$$

□

Proof. Using $f(t) := \mathbb{E}[e^{tS_n}] = \left(\frac{e^t+e^{-t}}{2}\right)^n$, $\mathbb{E}[S_n^4] = f^{(4)}(t)|_{t=0}$. We have

$$\mathbb{E}\left[\frac{S_n^4}{n^4}\right] = n^{-3} + 6C_n^2 n^{-4} = O(n^{-2}).$$

By Theorem 3.10, we complete the proof. □

Remark 3.13. *By Bernstein inequality,*

$$\mathbb{P}\left(\left|\frac{S_n}{n}\right| \geq \varepsilon\right) \leq 2\exp\left(-\frac{2n\varepsilon^2}{(1+2\varepsilon)^2}\right),$$

it is obvious that (2) *holds true.*

4 Between LLN and LIL

By (2), we have $|S_n| = o(n)$ *a.s.*, it is natural to ask whether a better rate can obtained, in fact we have.

Theorem 4.1. *For any* $\varepsilon > 0$,

$$\lim_{n\to\infty} \frac{S_n}{n^{\frac{1}{2}+\varepsilon}} = 0 \ a.s..$$

Proof. For any a position integer, by $\mathbb{E}[S_n^{2K}] = f^{(2K)}(t)|_{t=0}$, we have

$$\mathbb{E}[S_n^{2K}] = O(n^K).$$

Note that for $2\varepsilon K > 1$, we have

$$\mathbb{E}\left[\left|\frac{S_n^{2K}}{n^{K+2\varepsilon K}}\right|\right] \lesssim \frac{1}{n^{2\varepsilon K}}.$$

By Lemma 3.10, we complete the proof. □

By Borel-Cantelli lemma, we can obtain

Theorem 4.2.

$$\limsup_{n\to\infty} \frac{|S_n|}{n^{\frac{1}{2}} \log n} \le 1 \ a.s..$$

Proof. By $\mathbb{E}[e^{tS_n}] = \left(\frac{e^t+e^{-t}}{2}\right)^n$, we have

$$\mathbb{E}\left[\exp\left(n^{-\frac{1}{2}} S_n\right)\right] \to e^{1/2}.$$

Hence,

$$\mathbb{P}(S_n \ge (1+\varepsilon) n^{\frac{1}{2}} \log n) = \mathbb{P}\left(\exp(n^{-\frac{1}{2}} S_n) \ge n^{1+\varepsilon}\right) \lesssim \frac{1}{n^{1+\varepsilon}}$$

Moreover,

$$\limsup_{n\to\infty} \frac{S_n}{n^{\frac{1}{2}} \log n} \le 1 \ a.s..$$

By the symmetry of S_n i.e. S_n equal to $-S_n$ in law, we complete the proof. □

Theorem 4.3. *For any* $\varepsilon > 0$,

$$\lim_{n\to\infty} \frac{\mathbf{S}_n}{\sqrt{n(\log n)^{1+\varepsilon}}} = 0 \ a.s..$$

Proof. First, we prove Kolmogorov's maximal inequality. Let $X_1, X_2, \cdots$ be independent, mean-zero and $\mathbb{E}[X_k^2] < \infty \ \forall k \in \mathbb{N}_+$. Then

$$\mathbb{P}\left(\sup_{1\le k\le n} |S_k| > \lambda\right) \le \frac{\mathbb{E}[S_n^2]}{\lambda^2} = \frac{1}{\lambda^2}\sum_{k=1}^n \mathbf{Var}(X_k).$$

We partition $A^* := \{\sup_{1\le k\le n} S_n > \lambda\}$ into the events $A_k := \{|S_k| > \lambda \text{ and } |S_j| \le \lambda \text{ for all } j < k\}$, then we have

$$\begin{aligned}
\mathbb{E}[S_n^2] \ge \mathbb{E}[S_n^2\,\mathbf{1}_{A^*}] &= \sum_{k=1}^{n} \mathbb{E}[S_n^2\,\mathbf{1}_{A_k}] \\
&= \sum_{k=1}^{n} \left(\mathbb{E}[S_k^2\,\mathbf{1}_{A_k}] + 2\mathbb{E}[S_k(S_n - S_k)\,\mathbf{1}_{A_k}] + \mathbb{E}[(S_n - S_k)^2\,\mathbf{1}_{A_k}]\right) \\
&\ge \sum_{k=1}^{n} \mathbb{E}[S_k^2\,\mathbf{1}_{A_k}] \ge \sum_{k=1}^{n} \lambda^2 \mathbb{P}(A_k) = \lambda^2 \mathbb{P}(A^*).
\end{aligned}$$

Second, let $X_1, X_2, \cdots$ be independent, mean-zero and $\mathbb{E}[X_k^2] < \infty\ \forall k \in \mathbb{N}_+$, then

$$\sum_{i=1}^{\infty} \mathbf{Var}(X_i) < \infty \Rightarrow \sum_{i=1}^{\infty} X_i < \infty\ a.s..$$

By the assumptions about $\{X_i\}_{i\ge1}$, we see $\{S_n\}_{n\ge1}$ is a the Cauthy sequence in $L^2(\Omega)$ space. Therefore, there exist a $S_\infty \in L^2(\Omega)$ such that $S_n \to S_\infty$ in $L^2(\Omega)$. Moreover, there exist a subsequence $\{n_k\}_{k\ge1}$ such that $S_{n_k} \to S_\infty\ a.s...$

For any $k \ge 0$ (let $n_0 := 0, S_0 = 0$), by Kolmogorov inequality, we have

$$\mathbb{P}\left(\max_{n_k < p \le n_{k+1}} |S_p - S_{n_k}| \ge \varepsilon\right) \le \frac{1}{\varepsilon^2}\mathbb{E}[\left|S_{n_{k+1}} - S_{n_k}\right|^2].$$

Note that

$$\sum_{k=1}^{\infty} \mathbb{E}[\left|S_{n_{k+1}} - S_{n_k}\right|^2] = \sum_{n=1}^{\infty} \mathbb{E}[X_n^2] < \infty.$$

By Borel-Cantelli lemma, we obtain that

$$\max_{n_k < p \le n_{k+1}} |S_p - S_{n_k}| \to 0\ a.s..$$

Combining $S_{n_k} \to S_\infty\ a.s.$, we can obtain $S_n \to S_\infty\ a.s..$ which called **Extract Subsequence Method**. In fact, for n large enough, there exists a unique k large enough such that $n_{k-1} < n \le n_k$, then

$$|S_n - S_\infty| \le \left|S_{n_{k-1}} - S_\infty\right| + \max_{n_{k-1} < m \le n_k} \left|S_m - S_{n_{k-1}}\right| \to 0\ a.s..$$

Kronecker's lemma Let $\{a_n\}_{n\ge1}$ is a sequence of real number, and suppose $b_n \uparrow \infty$. If $\sum_i \frac{a_i}{b_i} < \infty$, then $\frac{\sum_{k=1}^n a_k}{b_n} \to 0$.

Finally, let $a_n = \mathbf{X}_n(\omega)$ and $b_n = \sqrt{n(\log n)^{1+\varepsilon}} \uparrow \infty$, it suffices to show that

$$\sum_{k=1}^{\infty} \frac{\mathbf{X}_k}{\sqrt{n(\log n)^{1+\varepsilon}}} < \infty\ a.s..$$

We only need to check

$$\sum_{i=1}^{\infty} \mathbf{Var}\left(\frac{\mathbf{X}_k}{\sqrt{n(\log n)^{1+\varepsilon}}}\right) = \sum_{i=1}^{\infty} \frac{\mathbf{Var}(\mathbf{X}_k)}{n(\log n)^{1+\varepsilon}} = \sum_{i=1}^{\infty} \frac{1}{n(\log n)^{1+\varepsilon}} < \infty.$$

Let $f(x) = \frac{1}{x(\log x)^\alpha}$, for x large enough, we have $f(x) > 0$ and f is a monotonically decreasing continuous function about x. Define $F(x) = \log\log x$ if $\alpha = 1$, $F(x) = \frac{1}{1-\alpha}(\log x)^{1-\alpha}$ if $\alpha \neq 1$, then we have $F'(x) = f(x)$ for x large enough. Therefore,

$$\int_N^\infty f(x)\,dx = \begin{cases} \infty, & \text{if } \alpha \geq 1, \\ \frac{1}{\alpha-1}(\log N)^{1-\alpha}, & \text{if } \alpha > 1. \end{cases}$$

□

Similarly, we can obtain that for any $\varepsilon > 0$, $k \in \mathbb{N}_+$

$$\lim_{n\to\infty} \frac{\mathbf{S}_n}{\sqrt{n \log n \log^{(2)} n \cdots (\log^{(k)} n)^{1+\varepsilon}}} = 0 \ a.s..$$

The best possible rate was obtained by Khinchine which is called Law of Iterated Logarithm,

$$\limsup_{n\to\infty} \frac{S_n}{\sqrt{2n \log\log n}} = 1 \ a.s..$$

Lemma 4.4. *For any positive integer N, we have*

$$\mathbb{P}\left(S_n \geq k\right) \leq e^{-\frac{k^2}{2n}}$$

Proof. Since

$$\mathbb{P}\left(S_n \geq k\right) \leq \frac{\mathbb{E}[e^{tS_n}]}{e^{tk}} = \frac{\left(\mathbb{E}[e^{tX_1}]\right)^n}{e^{tk}}$$

and

$$\mathbb{E}[e^{tX_1}] = \frac{e^t + e^{-t}}{2} \leq e^{\frac{t^2}{2}}.$$

By taking $t = \frac{k}{n}$, we have

$$\mathbb{P}\left(S_n \geq k\right) \leq \frac{e^{\frac{nt^2}{2}}}{e^{tk}} = e^{-\frac{k^2}{2n}}.$$

□

Lemma 4.5 (Reflection principle)**.** *For any positive integer m, we have*

$$\mathbb{P}(M_n^+ \geq m, S_n = s) = \begin{cases} \mathbb{P}(S_n = s), & \text{if } s \geq m, \\ \mathbb{P}(S_n = 2m - s), & \text{if } s < m, \end{cases}$$

and

$$\mathbb{P}(M_n^+ \geq m) = \mathbb{P}(S_n \geq m) + \sum_{s=-\infty}^{m-1} \mathbb{P}(S_n = 2m - s) = \mathbb{P}(S_n = m) + \sum_{k=m+1}^{\infty} 2\mathbb{P}(S_n = k)$$

and thus

$$\mathbb{P}(M_n^+ \geq m) = 2\mathbb{P}(S_n \geq m+1) + \mathbb{P}(S_n = m) \leq 2\mathbb{P}(S_n \geq m).$$

Lemma 4.6.

$$\mathbb{P}(S_n = k) \sim \frac{e^{-\frac{k^2}{2n}}}{\sqrt{\pi n}}.$$

Proof. Recall Stirling's approximation

$$n! \sim \sqrt{2\pi n}\left(\frac{n}{e}\right)^n$$

Then, we have

$$\begin{aligned}\mathbb{P}\left(S_{2n}=2k\right) &= \frac{(2n)!}{(n+k)!(n-k)!}2^{-2n} \\ &\sim \frac{1}{\sqrt{\pi n}}\frac{1}{\left(1+\frac{k}{n}\right)^{n+k}\left(1-\frac{k}{n}\right)^{n-k}}\frac{1}{\sqrt{\left(1+\frac{k}{n}\right)\left(1-\frac{k}{n}\right)}} \\ &= \frac{1}{\sqrt{\pi n}}\frac{\left(1-\frac{k}{n}\right)^k}{\left(1-\frac{k^2}{n^2}\right)^{n+\frac{1}{2}}\left(1+\frac{k}{n}\right)^k}\end{aligned}$$

Note that we need $(n-k)\to\infty$ to use Stirling's formula. We choose $k=\lfloor x\sqrt{\frac{n}{2}}\rfloor$ so that $\frac{2k}{\sqrt{2n}}\to x$. It is not hard to see that if $x_n\to 0$, and $y_n\to\infty$ such that $x_ny_n\to t$, then $(1+x_n)^{y_n}\to e^t$. Therefore,

$$\mathbb{P}(S_{2n}=2k)\sim\frac{1}{\sqrt{\pi n}}\left(1-\frac{x^2}{2n}\right)^{-n}\left(1-\frac{x}{\sqrt{2n}}\right)^{x\frac{\sqrt{n}}{\sqrt{2}}}\left(1+\frac{x}{\sqrt{2n}}\right)^{-x\frac{\sqrt{n}}{\sqrt{2}}}\to\frac{1}{\sqrt{\pi n}}e^{-\frac{x^2}{2}}.$$

□

Lemma 4.7. *Let $k>n^{\frac{1}{2}}$, then there exist a constant C such that*

$$\mathbb{P}(S_n\geq k)\geq C\cdot\frac{n^{\frac{1}{2}}}{k}e^{-\frac{k^2}{2n}}.$$

Proof. It is easy to see

$$\mathbb{P}(S_n\geq k)\geq\mathbb{P}(k\leq S_n\leq k+\frac{n}{k})\geq C\cdot n^{-\frac{1}{2}}\sum_{m=k}^{k+\frac{n}{k}}e^{-\frac{m^2}{2n}}.$$

For $k\leq m\leq k+\frac{n}{k}$, we have

$$e^{-\frac{k^2}{2n}}\geq e^{-\frac{m^2}{2n}}\geq e^{-\frac{(k+\frac{n}{k})^2}{2n}}=\exp\left(-\frac{k^2}{2n}-1-\frac{n}{2k^2}\right)\geq C\cdot e^{-\frac{k^2}{2n}}.$$

Therefore, we have

$$\mathbb{P}(S_n\geq k)\geq C\cdot\frac{n^{\frac{1}{2}}}{k}e^{-\frac{k^2}{2n}}.$$

□

Theorem 4.8. *Define $F_n:=\sqrt{2n\log\log n}$, then we have*

$$\limsup_{n\to\infty}\frac{S_n}{F_n}=1\ a.s..$$

Proof. The proof will be presented in two steps. The first one gives an upper bound of $\limsup_{n\to\infty}\frac{S_n}{F_n}$, i.e. for any $\varepsilon>0$, we show that

$$\limsup_{n\to\infty}\frac{S_n}{F_n}\leq 1+\varepsilon\ a.s..$$

The second one gives a lower bound of $\limsup_{n\to\infty} \frac{S_n}{F_n}$, i.e. for $0 < \varepsilon < \frac{1}{2}$,

$$\limsup_{n\to\infty} \frac{S_n}{F_n} \geq 1 - \varepsilon \ a.s..$$

Step 1. Let $\Theta > 1$, $n_k := \lfloor \Theta^k \rfloor$, by reflection principle and Lemma 4.4, we have

$$\begin{aligned}
\mathbb{P}\left(M_{n_k}^+ \geq (1+\varepsilon)F_{n_k}\right) &\leq 2\mathbb{P}\left(S_{n_k} \geq (1+\varepsilon)F_{n_k}\right) \\
&\leq 2\exp\left(-\frac{(1+\varepsilon)^2 F_{n_k}^2}{2n_k}\right) \\
&= 2\exp\left(-(1+\varepsilon)^2 \log\log n_k\right) \sim (k\log\Theta)^{-(1+\varepsilon)^2}
\end{aligned}$$

By Borel-Cantelli lemma, we have

$$M_{n_k}^+ \leq (1+\varepsilon)F_{n_k} \ a.s.,$$

for all but finitely many k.

Let $n_k \leq n < n_{k+1}$,

$$\frac{S_n}{F_n} \leq \frac{M_n^+}{F_n} = \frac{M_{n_{k+1}}^+}{F_{n_{k+1}}} \frac{F_{n_{k+1}}}{F_n} \frac{M_n^+}{M_{n_{k+1}}^+} \leq (1+\varepsilon)\frac{F_{n_{k+1}}}{F_n} \leq 1 + 2\varepsilon \ a.s.,$$

where $\Theta(\varepsilon, k)$ is close enough to 1.

Step 2. Define $\mathbf{n}_k = n_{k+1} - n_k$, by the definition of $\{S_n\}_{n\geq 1}$, we have $\{S_{n_{k+1}} - S_{n_k}\}_{k\geq 1}$ is mutually independent, $S_{\mathbf{n}_k}$ and $S_{n_{k+1}} - S_{n_k}$ are equal in law. By Lemma 4.6, for large $k(\varepsilon)$ enough, we have

$$\begin{aligned}
\mathbb{P}\left(S_{\mathbf{n}_k} = S_{n_{k+1}} - S_{n_k} \geq (1-\varepsilon)F_{\mathbf{n}_k}\right) &\geq C \cdot \frac{\mathbf{n}_k^{\frac{1}{2}}}{(1-\varepsilon)F_{\mathbf{n}_k}} \exp\left(-\frac{(1-\varepsilon)^2 F_{\mathbf{n}_k}^2}{2\mathbf{n}_k}\right) \\
&\sim \frac{1}{\sqrt{\log\log \mathbf{n}_k}} (\log \mathbf{n}_k)^{-(1-\varepsilon)^2}
\end{aligned}$$

where the last sim is due to $0 < \varepsilon < \frac{1}{2}$. Note that $\log \mathbf{n}_k \sim k$, we have $\sum_{k=1}^{\infty} \mathbb{P}(S_{\mathbf{n}_k} \geq (1-\varepsilon)F_{\mathbf{n}_k}) = \infty$, by Borel-Cantelli lemma, we have

$$S_{n_{k+1}} \geq S_{n_k} + (1-\varepsilon)F_{\mathbf{n}_k} \quad i.o.\ a.s..$$

By the symmetric of the upper bound, we have

$$\liminf_{k\to\infty} \frac{S_{n_k}}{F_{n_k}} \geq \liminf_{n\to\infty} \frac{S_n}{F_n} \geq -(1+\varepsilon) \ a.s..$$

Therefore, we have

$$\begin{aligned}
\frac{S_{n_{k+1}}}{F_{n_{k+1}}} &\geq \frac{S_{n_k}}{F_{n_k}} \frac{F_{n_k}}{F_{n_{k+1}}} + (1-\varepsilon)\frac{F_{\mathbf{n}_k}}{F_{n_{k+1}}} \\
&\geq -(1+\varepsilon)\frac{F_{n_k}}{F_{n_{k+1}}} + (1-\varepsilon)\frac{F_{\mathbf{n}_k}}{F_{n_{k+1}}} \to -\frac{(1+\varepsilon)}{\Theta^{\frac{1}{2}}} + (1-\varepsilon)\left(\frac{\Theta-1}{\Theta}\right)^{\frac{1}{2}}.
\end{aligned}$$

Take $\varepsilon \to 0^+$ and $\Theta \to \infty$, we complete the proof. □

Using almost the same method, we can prove the result about Brownian motion. For the convenience of the readers, we provide the detailed proof.

Theorem 4.9. *For a Brownian motion B in $\mathbb{R}$, we have*

$$\limsup_{t\to\infty} \frac{B_t}{\sqrt{2t\log\log t}} = 1 \ a.s..$$

Proof. when $u \to \infty$, we have

$$\int_u^\infty e^{-\frac{1}{2}}\,dx \sim u^{-1}\int_u^\infty xe^{-\frac{x^2}{2}} = u^{-1}e^{-\frac{u^2}{2}}.$$

In fact,

$$\int_u^\infty \frac{1}{x}(xe^{-\frac{x^2}{2}})\,dx = \int_u^\infty \frac{1}{x}d(e^{-\frac{x^2}{2}}) = \frac{1}{x}e^{-\frac{x^2}{2}}\Big|_0^\infty - \int_u^\infty \frac{1}{x^2}e^{-\frac{x^2}{2}}\,dx.$$

Let $M_t^+ = \sup_{0\le s\le t} B_s$, then by reflection principle, we have

$$\mathbb{P}(M_t^+ > ut^{\frac{1}{2}}) = 2\mathbb{P}(B_t > ut^{\frac{1}{2}}) \sim \frac{2}{\sqrt{2\pi}}u^{-1}e^{-\frac{u^2}{2}}.$$

Step 1. Define $F_t = \sqrt{2t\log\log t}$, for any $\Theta > 1$ and $1+\varepsilon > 1$, for n large enough, we have

$$\begin{aligned}
&\mathbb{P}(M_{\Theta^n}^+ > (1+\varepsilon)F_{\Theta^n}) \\
\le& 2\mathbb{P}\left(\frac{B_{\Theta^n}}{\sqrt{\Theta^n}} > \frac{(1+\varepsilon)F_{\Theta^n}}{\sqrt{\Theta^n}}\right) \\
\lesssim& \sqrt{\frac{\Theta^n}{(1+\varepsilon)^2F_{\Theta^n}^2}}\exp\left(-\frac{1}{2}\frac{(1+\varepsilon)^2F_{\Theta^n}^2}{\Theta^n}\right) \\
\lesssim& \frac{1}{\sqrt{\log\log\Theta^n}}\exp\left(-(1+\varepsilon)^2\log\log\Theta^n\right) \sim (n\log\Theta)^{-(1+\varepsilon)^2}.
\end{aligned}$$

By the Borel-Cantelli lemma, we obtain that for n large enough,

$$\frac{M_{\Theta^n}^+}{F_{\Theta^n}} \le (1+\varepsilon) \ a.s..$$

Therefore, for $\Theta^n \le t < \Theta^{n+1}$, Θ approach 1,

$$\frac{B_t}{F_t} \le \frac{M_t^+}{F_t} = \frac{M_{\Theta^{n+1}}^+}{F_{\Theta^{n+1}}}\frac{F_{\Theta^{n+1}}}{F_t}\frac{M_t^+}{M_{\Theta^{n+1}}^+} \le (1+\varepsilon)\frac{F_{\Theta^{n+1}}}{F_t} \le 1+2\varepsilon \ a.s..$$

Step 2. For $0 < \varepsilon < \frac{1}{2}$, we have

$$\begin{aligned}
\mathbb{P}\left(B_{\Theta^{n+1}} - B_{\Theta^n} > (1-\varepsilon)F_{[\Theta^{n+1}-\Theta^n]}\right) &\ge C\cdot\frac{(\Theta^{n+1}-\Theta^n)^{\frac{1}{2}}}{F_{[\Theta^{n+1}-\Theta^n]}}\exp\left(-\frac{(1-\varepsilon)^2F_{[\Theta^{n+1}-\Theta^n]}^2}{2(\Theta^{n+1}-\Theta^n)}\right) \\
&\sim \frac{1}{\sqrt{\log\log(\Theta^{n+1}-\Theta^n)}}\left(\log(\Theta^{n+1}-\Theta^n)\right)^{-(1-\varepsilon)^2}
\end{aligned}$$

Since $\log(\Theta^{n+1}-\Theta^n) \sim n$, we have

$$\sum_{n=1}^\infty \mathbb{P}\left(B_{\Theta^{n+1}} - B_{\Theta^n} > (1-\varepsilon)F_{[\Theta^{n+1}-\Theta^n]}\right) < \infty.$$

By Borel-Cantelli lemma, we have

$$B_{\Theta^{n+1}} \ge B_{\Theta^n} + (1-\varepsilon)F_{[\Theta^{n+1}-\Theta^n]} \quad i.o.\ a.s..$$

By the symmetric of the upper bound, we have

$$\liminf_{n\to\infty} \frac{B_{\Theta^n}}{F_{\Theta^n}} \geq \liminf_{t\to\infty} \frac{B_t}{F_t} \geq -(1+\varepsilon) \ a.s..$$

Therefore, we have

$$\frac{B_{\Theta^{n+1}}}{\Theta^{n+1}} \geq B_{\Theta^n} + (1-\varepsilon)F_{[\Theta^{n+1}-\Theta^n]} \quad i.o. \ a.s..$$

$$\begin{aligned}\frac{B_{\Theta^{n+1}}}{\Theta^{n+1}} &\geq \frac{B_{\Theta^n}}{F_{\Theta^n}}\frac{F_{\Theta^n}}{F_{\Theta^{n+1}}} + (1-\varepsilon)\frac{F_{[\Theta^{n+1}-\Theta^n]}}{F_{\Theta^{n+1}}} \\ &\geq -(1+\varepsilon)\frac{F_{\Theta^n}}{F_{\Theta^{n+1}}} + (1-\varepsilon)\frac{F_{[\Theta^{n+1}-\Theta^n]}}{F_{\Theta^{n+1}}} \to -\frac{(1+\varepsilon)}{\Theta^{\frac{1}{2}}} + (1-\varepsilon)\left(\frac{\Theta-1}{\Theta}\right)^{\frac{1}{2}}.\end{aligned}$$

Take $\varepsilon \to 0^+$ and $\Theta \to \infty$, we complete the proof. □